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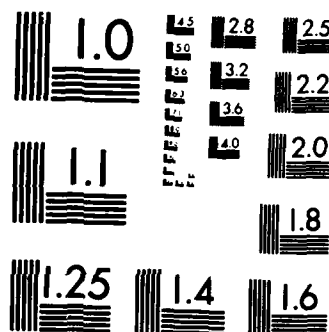
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DISTRIBUTION RESULTS FOR POSITIVE DEFINITE QUADRATIC  
FORMS WITH REPEATED ROOTS

BY

M. E. BOCK

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Distribution Results for Positive Definite Quadratic  
Forms with Repeated Roots

By

M.E. Bock

SECTION 1. INTRODUCTION

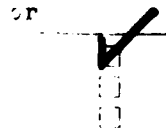
This paper examines the distribution functions of quadratic forms  
of the type

$$(*) \quad Q = \sum_{i=1}^k c_i \chi_{p_i}^2$$

where the  $c_i$  are distinct positive constants and the chi-square  
variables are independent with degrees of freedom  $p_i$  and where  
at least one of the  $p_i$ 's is larger than one. The  $c_i$  are the  
roots and the  $p_i$  their corresponding multiplicity.

Distributions of the form  $Q$  arise in a number of applica-  
tions. Solomon (1961) noted that probabilities of hitting targets  
frequently reduce to the distribution of quadratic forms of the  
type  $Q$ . Pillai and Young (1973) show that the trace of a Wishart  
matrix is distributed as  $Q$  with all the  $p_i$ 's equal. The variable  
 $Q^{\frac{1}{2}}$  arises in the engineering literature described as a weighted  
unbiased Rayleigh variate. (See Miller (1975).)

The form  $Q$  also arises in goodness-of-fit tests. Chernoff and  
Lehmann (1954) showed that the asymptotic distribution of certain



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chi-square goodness-of-fit tests has the form  $Q$ . The general results of Moore and Spruill (1975), Moore (1978) and LeCam, Mahan and Singh (1983) reveal that asymptotic distributions of the form  $Q$  are appropriate for an even wider variety of tests. Alvo, Cabilio and Feigen (1982) prove that the asymptotic distribution of the average Kendall tau statistic has the form  $Q$ .

Section 2 of this paper treats the general form  $Q$  given in (\*) with several  $p_i$  larger than one. It is shown that its distribution may be represented as a finite sum of the distributions of quadratic forms of the type  $Q$  with at most one  $p_i$  larger than one and with no more distinct roots than the original form. In particular, if at least two of the  $p_i$  are larger than one in the quadratic form  $Q$ , then

$$P[Q > c] = \sum_{n=1}^N \beta_n P[Q_n > c]$$

where each quadratic form  $Q_n$  has exactly one root with multiplicity greater than one. Furthermore, the number of distinct roots in each  $Q_n$  equals or is one more than the number of odd  $p_i$  in  $Q$ .

A general method is given for reducing the multiplicity  $p_i$  of a root  $c_i$  in  $Q$ : For  $p_i \geq 2$ ,

$$P[Q > x] = 2c_i f_Q(x) + P[Q_i > x]$$

where  $f_Q(x)$  is the density of  $Q$  and  $Q_i$  is the quadratic form which matches  $Q$  except that the multiplicity of  $c_i$  is  $(p_i - 2)$  rather than  $p_i$ .

In Section 3 the distribution function of  $Q$  is completely specified for the case of exactly two distinct roots  $c_1$  and  $c_2$ ,

i.e.,  $k$  is two in the above (\*) representation of  $Q$ . The distribution function is written in terms of commonly tabled functions and may be expressly evaluated. Section 4 relates the problem of finding the distribution of the trace of a  $2 \times 2$  Wishart matrix to the special case of the problem treated in Section 3 where  $p_1 = p_2$  in the quadratic form  $Q$ .

In Section 5, results are given for the quadratic form

$$Q_0^{(n)} = \chi_{2n}^2 + \sum_{i=1}^r c_i^{(i)} \chi_1^2$$

where  $0 < c_i < 1$ ,  $i = 1, \dots, r$ . This form arises in the asymptotic distribution of certain chi-square goodness-of-fit tests. The distribution function is completely specified in terms of tabled functions when  $n$  is 1. This is also true for the case when both  $n$  and  $r$  equal two in  $Q_0^{(n)}$ .

The distributional results obtained in this work increase the importance of tables for the distribution of  $Q$  as defined by (\*). Define  $P = \sum_{i=1}^k p_i$  in the definition of  $Q$ . Tables for the distribution function of  $Q$  for  $P = 2$  and  $3$  have been given by Grad and Solomon (1955) and Solomon (1960) and Marsaglia (1960). (An abridged version appears in Owen (1962).) Johnson and Kotz (1968) give tables for  $P = 4$  and  $5$ . Many of the formulas given in this paper reduce the distribution of a  $Q$  with a much larger value of  $P$  to finite sums involving the distribution of a  $Q$  with a much smaller value of  $P$ . Thus these tables of Solomon, and Grad and Solomon, and Marsaglia, and Johnson and Kotz become even more useful. It seems clear that the tables

of Solomon should appear in a more easily available location than the present technical report since so many of these formulas depend on them.

In the cases where the exact finite formulas given here become too involved or unwieldy, it is clearly appropriate to use approximations to the distribution function. Solomon and Stephens (1977,1978) have compared various methods and give references. Of particular interest are the ones to Kotz, Johnson and Boyd (1967), Imhof (1961) and Jensen and Solomon (1972) and Davis (1977). The formulas given in this work may offer insight for other approximations.

## SECTION 2

### RESULTS FOR REPEATED ROOTS IN THE GENERAL FORM

Let  $c_j$  be positive constants distinct and assume that the  $\chi_{p_j}^2$  are independent chi-square random variables with  $p_j$  degrees of freedom. Then define the quadratic form  $Q$  to be

$$Q = \sum_{j=1}^k c_j \chi_{p_j}^2.$$

The  $c_j$  are called the roots of the form  $Q$  and the  $p_j$  their multiplicity.

When the  $p_j$  are all even, Box (1954) has written the distribution of  $Q$  in the form

$$P[Q > c] = \sum_{j=1}^k \sum_{s=1}^{p_j/2} \alpha_{j,s} P[\chi_{2s}^2 > c/c_j]$$

where the  $\alpha_{j,s}$  are positive constants depending on  $\{c_i\}_{i=1}^k$  and  $\{p_i\}_{i=1}^k$ . A result of this paper shows that if two or more of the  $p_j$  are larger than one then the distribution of  $Q$  has the form

$$P[Q > c] = \sum_{n=1}^N \beta_n P[Q_n > c]$$

where each  $Q_n$  is a quadratic form with no more than one repeated root (i.e. at most one  $p_j$  is greater than one). Furthermore the roots of the  $Q_n$  are the  $c_j$ 's associated with the odd  $p_j$ 's in  $Q$  and possibly one  $c_j$  associated with an even  $p_j$ .



Also in this section formulas are given for reducing the distribution function of any quadratic form with a repeated root to that of a similar quadratic form with multiplicity smaller by two for that root. Recursive applications allow one to reduce a quadratic form distribution to that of a quadratic form with no repeated roots.

Theorem 2.1. Assume the chi-square random variables are independent and the roots  $c_i$  are positive and distinct in the quadratic form  $Q$  defined

$$Q = \sum_{i=1}^{k_0} c_i \cdot (1) \chi_1^2 + \sum_{i=1}^{k_1} c_{i+k_0} \cdot \chi_{1+2q_i}^2 + \sum_{i=1}^{k_2} c_{i+k_0+k_1} \cdot \chi_{2q_{i+k_1}}^2.$$

The  $q_i$ 's are all positive integers and it is assumed that  $k_1+k_2 > 1$ , and  $k_i \geq 0$ ,  $i = 0,1,2$ . Then  $P[Q > c]$  may be written as

$$P[Q > c] = \sum_{i=1}^{k_1+k_2} \sum_{\ell=1}^{q_i} \alpha_{i,\ell} P[Q_0 + c_{i+k_0} \cdot \chi_{2\ell}^2 > c]$$

where

$$Q_0 = \sum_{j=1}^{k_0+k_1} c_j \cdot (j) \chi_1^2.$$

The formulas for the constants  $\alpha_{i,\ell}$  are

$$\alpha_{i,q_i} = \prod_{\substack{n=k_0+1 \\ n \neq k_0+1}}^{k_0+k_1+k_2} \left( \frac{c_{i+k_0}}{c_{i+k_0}-c_n} \right)^{q_i}$$

and for  $\ell = 1, \dots, q_i-1$ ,

$$\alpha_{i, q_i - \ell} = \frac{\alpha_{i, q_i}}{\ell} \sum_{\substack{n=k_0+1 \\ n \neq i+k_0}}^{k_0+k_1+k_2} q_{n-k_0} \left( \frac{-c_n}{c_{i+k_0}-c_n} \right)^\ell$$

Proof. Define  $W = \sum_{i=1}^{k_0+k_1} c_i (1) \chi^2_1$  and  $V = \sum_{i=1}^{k_1+k_2} c_{i+k_0} \chi^2_{2q_i}$ . Then  $W+V$  has the same distribution as  $Q$  and

$$\begin{aligned} P[Q > c] &= P[W+V > c] \\ &= P[W > c] + P[W < c \text{ and } V > c-W] . \end{aligned}$$

Now  $P[W < c \text{ and } V > c-W] = E \left[ I(W < c) \cdot P[V > c-W|W] \right]$ . A theorem of Box [1954] implies that for  $W < c$ ,

$$\begin{aligned} P[V > c-W|W] \\ &= \sum_{i=1}^{k_1+k_2} \sum_{\ell=1}^{q_i} \alpha_{i,\ell} P[c_{k_0+i} \chi^2_{2\ell} > c-W|W] . \end{aligned}$$

Thus

$$\begin{aligned} P[W < c \text{ and } V > c-W] \\ &= \sum_{i=1}^{k_1+k_2} \sum_{\ell=1}^{q_i} \alpha_{i,\ell} E \left[ I(W < c) \cdot P[c_{k_0+i} \chi^2_{2\ell} > c-W|W] \right] \\ &= \sum_{i=1}^{k_1+k_2} \sum_{\ell=1}^{q_i} \alpha_{i,\ell} P[c_{k_0+i} \chi^2_{2\ell} > c-W \text{ and } W < c] . \end{aligned}$$

Because  $P[c_{k_0+i} \chi^2_{2\ell} > c-W \text{ and } W < c] = P[Q_{i,\ell} > c] - P[W > c]$ ,

where

$$Q_{i,l} = Q_0 + c_{i+k_0} \cdot \chi_{2l}^2,$$

we have

$$P[Q > c] = \left( 1 - \sum_{i=1}^{k_1+k_2} \sum_{l=1}^{q_i} \alpha_{i,l} \right) P[W > c] \\ + \sum_{i=1}^{k_1+k_2} \sum_{l=1}^{q_i} \alpha_{i,l} P[Q_{i,l} > c].$$

As noted by Pillai and Young (1973)

$$\sum_{i=1}^{k_1+k_2} \sum_{l=1}^{q_i} \alpha_{i,l} = 1.$$

Thus,

$$P[Q > c] = \sum_{i=1}^{k_1+k_2} \sum_{l=1}^{q_i} \alpha_{i,l} P[Q_{i,l} > c]. \quad ||$$

**Theorem 2.2.** Let  $W$  be a continuous nonnegative random variable and assume  $\chi_n^2$  has a central chi-square ( $n$ ) distribution independent of  $W$ . Let  $f_{Q_n}(x)$  be the density of  $Q_n = W + c_0 \chi_n^2$  where  $c_0 > 0$ . For  $c > 0$ , and  $n > 2$ ,

$$P[W + c_0 \chi_n^2 > c] = (2c_0) f_{Q_n}(c) + P[W + c_0 \chi_{n-2}^2 > c].$$

If  $n = 2$ , this is

$$P[W + c_0 \chi_2^2 > c] = (2c_0) f_{Q_2}(c) + P[W > c].$$

Corollary 2.3. For the quadratic form

$$Q = \sum_{i=0}^n c_i \chi_{p_i}^2$$

where the  $\chi_{p_i}^2$  are independent and distinct  $c_i > 0$ ,  $i = 0, \dots, n$ , and  $p_0 \geq 2$ ,

$$P[Q > c] = P[c_0 \chi_{p_0-2}^2 + \sum_{i=1}^n c_i \chi_{p_i}^2 > c] + 2f_Q(c)$$

where  $f_Q(x)$  is the density of  $Q$  and  $\chi_0^2 \equiv 0$ .

Proof of Theorem 2.2. Let  $Q_n = W + c_0 \chi_n^2$ . Let  $f_{Q_n}(x)$  be the density of  $Q_n$ . Then

$$f_{Q_n}(c) = \frac{d}{dc} [P[W + c_0 \chi_n^2 < c]] .$$

We may write

$$\begin{aligned} P[W + c_0 \chi_n^2 < c] &= \int_0^{c/c_0} \frac{u^{n/2-1} e^{-u/2}}{\Gamma(\frac{n}{2}) 2^{n/2}} \left[ \int_0^{c-c_0 u} dF_W \right] du \\ &= \int_0^c \frac{(\frac{c-t}{c_0})^{n/2-1}}{c_0} \frac{e^{-1/2(c-t)/c_0}}{\Gamma(\frac{n}{2}) 2^{n/2}} \left[ \int_0^t dF_W \right] dt \end{aligned}$$

(where  $t = c - c_0 u$  is the change of variable).

Differentiating this last expression implies

$$f_{Q_n}(c) = (2c_0)^{-1} \left\{ I(n \geq 3) \int_0^c \frac{(\frac{c-t}{c_0})^{\frac{n}{2}-2} e^{-(c-t)/2c_0}}{c_0 \Gamma(\frac{n-2}{2}) 2^{n/2-1}} \left[ \int_0^t dF_W \right] dt \right.$$

$$\begin{aligned}
& + I(n=2) \left\{ \int_0^c dF_W \right\} + (-1) \int_0^c \frac{\left(\frac{c-t}{c_0}\right)^{n/2-1} e^{-(c-t)/2c_0}}{c_0 \Gamma(\frac{n}{2}) 2^{n/2}} \left\{ \int_0^t dF_W \right\} dt \Bigg\} \\
& = (2c_0)^{-1} \{ I(n \geq 3) P[W+c_0 \chi_{n-2}^2 < c] + I(n=2) P[W < c] - P[W+c_0 \chi_n^2 < c] \} .
\end{aligned}$$

Thus, if  $n = 2$

$$P[W+c_0 \chi_2^2 > c] = P[W > c] + (2c_0) f_{Q_2}(c) .$$

If  $n \geq 3$ ,

$$P[W+c_0 \chi_n^2 > c] = P[W+c_0 \chi_{n-2}^2 > c] + (2c_0) f_{Q_n}(c) . \quad ||$$

Corollary 2.4. Define

$$Q_0 = c_0 \chi_{p_0}^2 + \sum_{i=1}^n c_i^{(i)} \chi_1^2$$

where  $c_i > 0$  and the chi-square variables are independent and  $p_0 \geq 2$ .

Let

$$Q_j = c_0 \chi_j^2 + \sum_{i=1}^n c_i^{(i)} \chi_1^2$$

with density  $f_{Q_j}(x)$ .

(a) For odd  $p_0$ ,

$$P[Q_0 > c] = P[c_0^{(0)} \chi_1^2 + \sum_{i=1}^n c_i^{(i)} \chi_1^2 > c] + (2c_0) \sum_{j=1}^{\frac{p_0-1}{2}} f_{Q_{1+2j}}(c) .$$

(b) For even  $p_0$ ,

$$P[Q_0 > c] = P\left[\sum_{i=1}^n c_i^{(1)} \chi_1^2 > c\right] + (2c_0)^{\frac{p_0}{2}} \sum_{j=1}^{\frac{p_0}{2}} f_{Q_{2j}}(c).$$

Proof. A recursive application of Corollary 2.3 gives the result. ||

Substitute the result of Corollary 2.4 in Theorem 2.1 to obtain the next result, recalling that  $\sum_1 \sum_\ell \alpha_{i,\ell} = 1$ :

Corollary 2.5. Define  $Q, Q_0, \alpha_{i,\ell}$  and  $c_i$  as in Theorem 2.1. Then if  $f_{i,\ell}(y)$  is the density for the quadratic form  $Q_0 + c_{i+k_0} \chi_{2\ell}^2$ , we have

$$P[Q > c] = P[Q_0 > c] + \sum_{i=1}^{k_1+k_2} \sum_{\ell=1}^{q_i} \beta_{i,\ell} \sum_{j=1}^{\ell} f_{i,j}(c)$$

where  $\beta_{i,\ell} = 2c_{i+k_0} \alpha_{i,\ell}$ .

Theorem 2.6. Let  $\{c_i\}_{i=0}^n$  be distinct and positive and let the chi-square variables be independent. For  $c > 0$  and  $c_0 > c_i$ ,  $i = 1, \dots, n$ ,

$$\begin{aligned} P\left[\sum_{i=1}^n c_i \chi_{p_i}^2 + c_0 \chi_2^2 > c\right] &= P\left[\sum_{i=1}^n c_i \chi_{p_i}^2 > c\right] + \\ &+ P\left[\sum_{i=1}^n c_i^* \chi_{p_i}^2 < c\right] e^{-c/2c_0} \prod_{i=1}^n \left(\frac{c_i^*}{c_i}\right)^{p_i/2} \end{aligned}$$

where  $c_i^* = (c_i^{-1} - c_0^{-1})^{-1}$ .

Proof.

$$P\left[\sum_{i=1}^n c_i \chi_{p_i}^2 + c_0 \chi_2^2 > c\right] = P\left[\sum_{i=1}^n c_i \chi_{p_i}^2 > c\right] + \\ + P\left[\sum_{i=1}^n c_i \chi_{p_i}^2 < c \text{ and } c_0 \chi_2^2 > c - \sum_{i=1}^n c_i \chi_{p_i}^2\right].$$

Thus

$$(*) = P\left[\sum_{i=1}^n c_i \chi_{p_i}^2 + c_0 \chi_2^2 > c\right] - P\left[\sum_{i=1}^n c_i \chi_{p_i}^2 > c\right] \\ = E\left[P[c_0 \chi_2^2 > c - \sum_{i=1}^n c_i \chi_{p_i}^2 \mid \sum_{i=1}^n c_i \chi_{p_i}^2 < c]\right] \\ = E\left[I\left(\sum_{i=1}^n c_i \chi_{p_i}^2 < c\right) e^{-(1/2 c_0)(c - \sum_{i=1}^n c_i \chi_{p_i}^2)}\right] \\ = e^{-c/2 c_0} E\left[e^{1/2 \sum_{i=1}^n (c_i/c_0) \chi_{p_i}^2} I\left(\sum_{i=1}^n c_i \chi_{p_i}^2 < c\right)\right].$$

Apply Lemma 4 of the Appendix  $n$  times to the last expectation gives

$$(*) = e^{-c/2 c_0} \left( \prod_{i=1}^n \left(1 - \frac{c_i}{c_0}\right)^{-p_i/2} \right) E\left[I\left(\sum_{i=1}^n \frac{c_i}{(1 - \frac{c_i}{c_0})} \chi_{p_i}^2 < c\right)\right]. \quad \parallel$$

### SECTION 3

#### QUADRATIC FORMS WITH TWO DISTINCT ROOTS

In this section we consider quadratic forms

$$Q = c_1 \chi_m^2 + c_2 \chi_n^2$$

where the  $c_1 > 0$  are distinct and the chi-square variables are independent.

The density of  $Q$  is given in Theorem 3.1 in terms of a confluent hypergeometric function. (It is given in Miller (1975, page 61) in essentially the same form.) Simple representations for the confluent hypergeometric function whose first two arguments are integers divided by two are given in Bock, Judge and Yancy (1984) and may be used to evaluate the density.

Finite representations for  $P[Q > c]$  are given in terms of commonly tabled special functions. In the case that both  $m$  and  $n$  are even, the result of Box (1954) given in section 2 shows that the special functions are probabilities of chi-square variables with even degrees of freedom. (There are simple formulas for these given in terms of the exponential function and a polynomial.) Theorem 3.3 provides a finite representation when  $m$  is odd and  $n$  is even. If  $c_2 > c_1$ , the special functions are probabilities of chi-square variables with odd degrees of freedom. If  $c_1 > c_2$  the special function is Dawson's integral tabled in Abramowitz and Stegun (1964). For odd  $m$  and  $n$ , Theorem 3.5 implies



that  $P[Q > c]$  may be written as  $P[c_1^{(1)} \chi_1^2 + c_2^{(2)} \chi_1^2 > c]$  (tabled by Solomon (1960)) plus a finite number of modified Bessel functions (tabled by Abramowitz and Stegun (1964)). Furthermore, Ruben (1960) notes the (unpublished) result of Dr. David Kleinecke that

$$P[c_1^{(1)} \chi_1^2 + c_2^{(2)} \chi_1^2 < c] = P[\chi_{2,A}^2 < B] - P[\chi_{2,B}^2 < A]$$

where

$$A = \frac{c}{4}(c_1^{-1/2} - c_2^{-1/2})^2, \quad B = \frac{c}{4}(c_1^{-1/2} + c_2^{-1/2})^2$$

and  $\chi_{2,\lambda}^2$  is a noncentral chi-square random variable with two degrees of freedom. Extensive tables of noncentral chi-square probabilities by Haynam, Govindarajulu, Leone and Siefert (1983) and these may be used to supplement the tables of Solomon (1960) and Marsaglia (1960).

**Theorem 3.1.** Let  $m$  and  $n$  be positive integers and let  $c_1, c_2$  and  $c$  be positive. Then the density of

$$Q = c_1 \chi_m^2 + c_2 \chi_n^2$$

is

$$f_Q(y) = \frac{y^{(m+n)/2-1} e^{-y/2c_1}}{\Gamma(\frac{m+n}{2}) (2c_1)^{m/2} (2c_2)^{n/2}} {}_1F_1\left(\frac{n}{2}, \frac{m+n}{2}; (c_1^{-1} - c_2^{-1}) \frac{y}{2}\right)$$

for  $y \geq 0$  where  $\chi_m^2$  and  $\chi_n^2$  are independent chi-square random variables.

Proof. Let  $W_1$  and  $W_2$  be independent random variables such that  $W_1/c_1$  has a chi-square ( $m$ ) distribution and  $W_2/c_2$  has a chi-square ( $n$ ) distribution. Then the density of  $W_1$  is

$$h_1(x) = \frac{x^{m/2-1} e^{-x/2c_1}}{\Gamma(\frac{m}{2}) (2c_1)^{m/2}}$$

for  $x \geq 0$ .

The density of  $W_2$  is

$$h_2(x) = \frac{x^{n/2-1} e^{-x/2c_2}}{\Gamma(\frac{n}{2}) (2c_2)^{n/2}}.$$

for  $x \geq 0$ .

Then the density of  $Q = W_1 + W_2$  is

$$f_Q(y) = \int_0^y h_1(y-x) h_2(x) dx$$

$$= \frac{e^{-y/2c_1} \int_0^y (y-x)^{m/2-1} x^{n/2-1} e^{-x/2(c_2^{-1}-c_1^{-1})} dx}{(2c_1)^{m/2} (2c_2)^{n/2} \Gamma(\frac{n}{2}) \Gamma(\frac{m}{2})}.$$

The integral in parentheses can be written as

$$\frac{\Gamma(\frac{n}{2}) \Gamma(\frac{m}{2}) y^{(m+n)/2-1}}{\Gamma(\frac{m+n}{2})} {}_1F_1\left(\frac{n}{2}, \frac{m+n}{2}; (c_1^{-1}-c_2^{-1}) \frac{y}{2}\right)$$

according to Lemma 1 of the Appendix. Thus for  $y \geq 0$ ,

$$f_Q(y) = \frac{e^{-y/2c_1} y^{(m+n)/2-1} {}_1F_1\left(\frac{n}{2}, \frac{m+n}{2}; (c_1^{-1}-c_2^{-1}) \frac{y}{2}\right)}{\Gamma\left(\frac{m+n}{2}\right) (2c_1)^{m/2} (2c_2)^{n/2}} . \quad ||$$

The following is a direct result of Corollary 2.3 and Theorem 3.1.

Corollary 3.2. Let  $c_0, c_1$  and  $c$  be positive and assume that chi-square variables are independent in the following expressions. Then if  $m > 2$ ,

$$P[c_0 \chi_m^2 + c_1 \chi_n^2 > c] = P[c_0 \chi_{m-2}^2 + c_1 \chi_n^2 > c] +$$

$$\frac{c_0 \left(\frac{c}{2}\right)^{(m+n)/2-1} e^{-c/2c_0}}{\Gamma\left(\frac{m+n}{2}\right) c_0^{m/2} c_0^{n/2}} {}_1F_1\left(\frac{n}{2}, \frac{m+n}{2}; \frac{c}{2}(c_0^{-1}-c_1^{-1})\right) .$$

For  $m = 2$ ,

$$P[c_0 \chi_2^2 + c_1 \chi_n^2 > c] = P[\chi_n^2 > \frac{c}{c_1}] +$$

$$\frac{\left(\frac{c}{2c_1}\right)^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} e^{-c/2c_0} {}_1F_1\left(\frac{n}{2}, \frac{n}{2}+1; \frac{c}{2}(c_0^{-1}-c_1^{-1})\right) .$$

Theorem 3.3. Let  $m$  be an odd integer and let  $n$  be an even integer. Then for positive  $c_0$  and  $c_1$  and  $c$ , and independent chi-square variables,

$$P[c_0 \chi_m^2 + c_1 \chi_n^2 > c] = P[\chi_m^2 > \frac{c}{c_0}] + (*) .$$

If  $c_1 > c_0$ , then  $x = \frac{c}{2} (c_1^{-1} - c_0^{-1}) < 0$  and

$$(*) = \frac{1}{\Gamma(\frac{m}{2})} \left(\frac{-c}{2c_0x}\right)^{m/2} \sum_{\ell=0}^{n/2-1} P[\chi_{m+2\ell}^2 < -2x] \frac{\Gamma(\ell + \frac{m}{2})}{\ell! x^\ell} P[\chi_{n-2\ell}^2 > \frac{c}{2c_1}] .$$

(The above also holds if  $m$  is even.) If  $c_1 < c_0$ , then

$$x = \frac{c}{2} (c_1^{-1} - c_0^{-1}) > 0 \text{ and}$$

$$(*) = e^x \sum_{\ell=0}^{n/2-1} P[\chi_{n-2\ell}^2 > \frac{c}{c_1}] \cdot \alpha_\ell \cdot \left\{ \frac{\mathcal{D}(x^{1/2})}{\sqrt{\pi/4}} - \sum_{t=0}^{\ell+(m-3)/2} \frac{(-1)^t x^{t+1/2}}{\Gamma(t + \frac{3}{2})} \right\}$$

where  $\mathcal{D}(y)$  is Dawson's integral and

$$\alpha_\ell = \frac{(-1)^{(m-1)/2} \Gamma(\frac{m}{2} + \ell)}{\Gamma(\frac{m}{2}) \ell!} \left(\frac{c}{2c_1x}\right)^\ell \left(\frac{c}{2c_0x}\right)^{m/2} .$$

Proof. Let  $m$  be an odd integer and  $n$  an even integer.

$$\begin{aligned} & P[c_0 \chi_m^2 + c_1 \chi_n^2 > c] \\ &= P[\chi_m^2 > \frac{c}{c_0}] + P[\chi_m^2 < \frac{c}{c_0} \text{ and } \chi_n^2 > \frac{c - c_0 \chi_m^2}{c_1}] . \end{aligned}$$

The last probability in the sum is

$$(*) = \int_0^{c/c_0} \frac{u^{m/2-1} e^{-u/2}}{\Gamma(\frac{m}{2}) 2^{m/2}} \left\{ \int_{(c-c_0u)/c_1}^{\infty} \frac{x^{n/2-1} e^{-x/2} dx}{\Gamma(\frac{n}{2}) 2^{n/2}} \right\} du .$$

The inner integral above in brackets is

$$e^{-(c-c_0u)/2c_1} \sum_{j=0}^{n/2-1} \frac{\left(\frac{c-c_0u}{2c_1}\right)^j}{j!} .$$

Using this and making the change of variable  $w = c_0 u / c$  in (\*) implies that

$$\begin{aligned}
 (*) &= \frac{e^{-c/2c_1}}{\Gamma(\frac{m}{2})} \left(\frac{c}{2c_0}\right)^{m/2} \sum_{j=0}^{n/2-1} \frac{\left(\frac{c}{2c_1}\right)^j}{j!} \int_0^1 w^{m/2-1} (1-w)^j e^{-wc(c_0^{-1}-c_1^{-1})/2} dw \\
 &= \frac{e^{-c/2c_1}}{\Gamma(\frac{m}{2})} \left(\frac{c}{2c_0}\right)^{m/2} \sum_{j=0}^{n/2-1} \frac{\left(\frac{c}{2c_1}\right)^j}{j!} [\beta(\frac{m}{2}, j+1) {}_1F_1(\frac{m}{2}, j+1 + \frac{m}{2}; \frac{c}{2}(c_1^{-1}-c_0^{-1}))].
 \end{aligned}$$

The evaluation of the integral follows from equation 3.393, #1, p. 318, of Gradshteyn and Ryzhik. Thus

$$(*) = \frac{e^{-c/2c_1} \left(\frac{c}{2c_0}\right)^{m/2}}{\Gamma(\frac{m}{2})} \sum_{j=0}^{n/2-1} \left(\frac{c}{2c_1}\right)^j \frac{\Gamma(\frac{m}{2})}{\Gamma(\frac{m}{2} + j + 1)} {}_1F_1(\frac{m}{2}, j+1 + \frac{m}{2}; \frac{c}{2}(c_1^{-1}-c_0^{-1})).$$

If  $c_0 > c_1$ , then  $x = \frac{c}{2}(c_1^{-1}-c_0^{-1}) < 0$  and Case 2' of Theorem 2 (in Remark) of Bock, Judge and Yancey (1984) implies that

$$\Gamma(\frac{m}{2}) {}_1F_1\left(\frac{m}{2}, j+1 + \frac{m}{2}; x\right) = \frac{(-1)^{(m-1)/2}}{(-x)^{1/2}} \sum_{k=(m+1)/2}^{j+(m+1)/2} \frac{\Gamma(k - \frac{1}{2}) x^{-(k-1)} P[\chi_{2k-1}^2 < -2x]}{(k - \frac{(m+1)}{2})! (j + \frac{(m+1)}{2} - k)!}.$$

Interchanging the summations in (\*) we may write for  $\ell = k - \frac{(m+1)}{2}$

$$(*) = \frac{e^{-c/2c_1}}{\Gamma(\frac{m}{2})} \left(\frac{c}{2c_0}\right)^{m/2} \sum_{\ell=0}^{n/2-1} \frac{\Gamma(\ell + \frac{m}{2})}{\ell! x^\ell} P[\chi_{m+2\ell}^2 < -2x] \cdot \sum_{j=\ell}^{n/2-1} \frac{\left(\frac{c}{2c_1}\right)^j}{(j-\ell)!}.$$

Note that  $\sum_{j=\ell}^{n/2-1} \frac{(\frac{c}{2c_1})^j}{(j-\ell)!} e^{\frac{-c}{2c_1}}$  is  $P[\chi_{n-2\ell}^2 > \frac{c}{c_1}]$ . If  $c_0 > c_1$ , then  $x = \frac{c}{2}(c_1^{-1} - c_0^{-1}) > 0$  and by Case 2 of Theorem 2 of Bock, Judge and Yancey (1984), implies

$$\begin{aligned} & \frac{\Gamma(\frac{m}{2})}{\Gamma(\frac{m}{2} + j + 1)} {}_1F_1(\frac{m}{2}, j + 1 + \frac{m}{2}; x) = e^x (-1)^{(m-1)/2} \cdot \\ & \cdot \sum_{s=0}^j \frac{x^{s+1/2-j-m/2}}{s!} \frac{\Gamma(j + \frac{m}{2} - s)}{(j-s)!} \left\{ \frac{2\mathcal{D}(x^{1/2})}{\sqrt{\pi x}} - \sum_{t=0}^{(m-3)/2+j-s} \frac{(-x)^t}{\Gamma(t + \frac{3}{2})} \right\} \\ & = e^x (-1)^{(m-1)/2} \sum_{\ell=0}^j x^{-\ell-(m-1)/2} \frac{\Gamma(\ell + \frac{m}{2})}{(j-\ell)! \ell!} \left\{ \frac{2\mathcal{D}(x^{1/2})}{\sqrt{\pi x}} - \sum_{t=0}^{\ell+(m-3)/2} \frac{(-x)^t}{\Gamma(t + \frac{3}{2})} \right\} \end{aligned}$$

where  $\mathcal{D}(y)$  is Dawson's integral, tabled in Abramowitz and Stegun.

Thus

$$\begin{aligned} (*) &= \frac{e^{-c/2c_0}}{\sqrt{\frac{c}{2c_0}}} \frac{(-\frac{c}{2c_0})^{(m+1)/2}}{\Gamma(\frac{m}{2})} (-1)^{\sum_{j=0}^{n/2-1} (\frac{c}{2c_1})^j} \sum_{\ell=0}^j \frac{x^{-\ell-(m-1)/2} \Gamma(\ell + \frac{m}{2})}{\ell! (j-\ell)!} \cdot \\ & \cdot \left\{ \frac{2\mathcal{D}(x^{1/2})}{\sqrt{\pi x}} - \sum_{t=0}^{\ell+(m-3)/2} \frac{(-x)^t}{\Gamma(t + \frac{3}{2})} \right\} \\ &= \frac{e^{-c/2c_0} (\frac{c}{2c_0})^{m/2}}{(-1)^{(m-1)/2} \Gamma(\frac{m}{2})} \left\{ \frac{2\mathcal{D}(x^{1/2})}{\sqrt{\pi x}} \left\{ \sum_{j=0}^{n/2-1} (\frac{c}{2c_1})^j \sum_{\ell=0}^j \frac{\Gamma(\ell + \frac{m}{2})}{\ell! (j-\ell)! x^{\ell+(m-1)/2}} \right\} \right. \\ & \left. - \left\{ \sum_{j=0}^{n/2-1} (\frac{c}{2c_1})^j \sum_{\ell=0}^j \frac{\Gamma(\ell + \frac{m}{2})}{\ell! (j-\ell)! x^{\ell+(m-1)/2}} \sum_{t=0}^{\ell+(m-3)/2} \frac{(-x)^t}{\Gamma(t + \frac{3}{2})} \right\} \right\} . \end{aligned}$$

By interchanging the orders of summation the subtracted bracketed expression with three summations may be written as

$$\begin{aligned}
 &= \sum_{\ell=0}^{n/2-1} \frac{\left(\frac{c}{2c_1}\right)^\ell \Gamma\left(\ell + \frac{m}{2}\right)}{\ell! x^{\ell+(m-1)/2}} \sum_{t=0}^{\ell+(m-3)/2} \frac{(-x)^t}{\Gamma\left(t + \frac{3}{2}\right)} \\
 &\quad \cdot \sum_{j=\ell}^{n/2-1} \frac{\left(\frac{c}{2c_1}\right)^{j-\ell}}{(j-\ell)!} \\
 &= \frac{e^{c/2c_1}}{x^{m/2}} \sum_{\ell=0}^{n/2-1} \frac{\Gamma\left(\ell + \frac{m}{2}\right)}{\ell!} \left(\frac{c}{2c_1 x}\right)^\ell \sum_{t=0}^{\ell+(m-3)/2} \frac{(-1)^t x^{t+1/2}}{\Gamma\left(t + \frac{3}{2}\right)} \\
 &\quad \cdot P[\chi_{n-2\ell}^2 > \frac{c}{c_1}] .
 \end{aligned}$$

A similar interchange in the bracketed expression multiplying  $\mathcal{D}(x^{1/2})$  implies

$$\begin{aligned}
 (*) &= \frac{e^{x/2} (-1)^{(m-1)/2}}{\Gamma\left(\frac{m}{2}\right)} \left(\frac{c_1}{c_2}\right)^{m/2} \left\{ \frac{\mathcal{D}(x^{1/2})}{\sqrt{\pi/4}} \sum_{\ell=0}^{n/2-1} \right. \\
 &\quad \left. \frac{\Gamma\left(\ell + \frac{m}{2}\right)}{\ell!} \left(\frac{c}{2c_1 x}\right)^{\ell+m/2} P[\chi_{n-2\ell}^2 > \frac{c}{c_1}] \right\} +
 \end{aligned}$$

$$- \left\{ \sum_{\ell=0}^{n/2-1} \frac{\Gamma(\ell + \frac{m}{2})}{\ell!} \left(\frac{c}{2c_1 x}\right)^{\ell+m/2} P[\chi_{n-2\ell}^2 > \frac{c}{c_1}] \cdot \right. \\ \left. \cdot \sum_{t=0}^{\ell+(m-3)/2} \frac{(-1)^t x^{t+1/2}}{\Gamma(t + \frac{3}{2})} \right\}.$$

Thus

$$(*) = e^x \cdot \sum_{\ell=0}^{n/2-1} P[\chi_{n-2\ell}^2 > \frac{c}{c_1}] \cdot \alpha_{\ell}$$

$$\left\{ \frac{D(x^{1/2})}{\sqrt{\pi/4}} - \sum_{t=0}^{\ell+(m-3)/2} \frac{(-1)^t x^{t+1/2}}{\Gamma(t + \frac{3}{2})} \right\}$$

where

$$\alpha_{\ell} = \frac{\Gamma(\frac{m}{2} + \ell) (-1)^{(m-1)/2}}{\Gamma(\frac{m}{2})!} \left(\frac{c}{2c_1 x}\right)^{\ell} \left(\frac{c}{2c_0 x}\right)^{m/2} \quad \parallel$$

Remark: (Following Theorem 3.3) Kummer's transform ( ${}_1F_1(a, b; c) = e^c {}_1F_1(b-a, b; -c)$ ) implies that (\*) may be written as

$$(*) = \frac{e^{-c/2c_0}}{\Gamma(\frac{m}{2} + 1)} \left(\frac{c}{2c_0}\right)^{m/2} \left\{ \sum_{j=0}^{n/2-1} \frac{(\frac{c}{2c_1})^j}{j!} \frac{\Gamma(j+1)\Gamma(1+\frac{m}{2})}{\Gamma(1)\Gamma(j+1+\frac{m}{2})} \cdot \right. \\ \left. \cdot {}_1F_1(1+j, 1+\frac{m}{2}+j; \frac{c}{2} (c_0^{-1} - c_1^{-1})) \right\}.$$

Note that the summation in the bracket represents the first  $\frac{n}{2}$  terms of an infinite series expansion for  ${}_1F_1(1, 1+\frac{m}{2}; \frac{c}{2c_0})$  given by Slater (1960) equation (2.3.2), p. 22. Thus as  $n \rightarrow \infty$ , the sum becomes  ${}_1F_1(1, 1+\frac{m}{2}; \frac{c}{2c_0})$ .



Examples (for Theorem 3.3). Let  $m$  be an odd integer and  $c_1 > c_0$ .

(a) For  $c_1 > c_0$ ,

$$P[c_0 \chi_m^2 + c_1 \chi_2^2 > c] = P[\chi_m^2 > \frac{c}{c_0}] + e^{-c/2c_1} \left(\frac{c_1}{c_1 - c_0}\right)^{m/2} P[\chi_m^2 < c(c_0^{-1} - c_1^{-1})] .$$

(b) For  $c_1 > c_0$ ,

$$P[c_0 \chi_m^2 + c_1 \chi_4^2 > c] = P[\chi_m^2 > \frac{c}{c_0}] + e^{-c/2c_1} \left(\frac{c_1}{c_1 - c_0}\right)^{m/2} \{P[\chi_m^2 < c(c_0^{-1} - c_1^{-1})](1 + \frac{c}{2c_1}) + P[\chi_{m+2}^2 < c(c_0^{-1} - c_1^{-1})] \frac{m}{2} \frac{c_0}{(c_0 - c_1)}\} .$$

(c) For  $c_1 < c_0$  and  $x = \frac{c}{2} (c_1^{-1} - c_0^{-1})$ ,

$$P[c_0 \chi_m^2 + c_1 \chi_2^2 > c] = P[\chi_m^2 > \frac{c}{c_0}] + e^{-c/2c_0} \left(\frac{-c}{2c_0 x}\right)^{(m-1)/2} \left\{ \frac{\mathcal{D}(x^{1/2})}{\sqrt{\pi x/4}} - \sum_{t=0}^{(m-3)/2} \frac{(-1)^t x^t}{\Gamma(t + \frac{3}{2})} \right\} .$$

Corollary 3.4. For independent chi-square variables and positive  $c_0, c_1$  and  $c$ , and  $x = \frac{c}{2} (c_1^{-1} - c_0^{-1})$

(a) if  $c_1 > c_0$ , then  $x < 0$  and

$$P[c_0 X_1^2 + c_1 X_{2k}^2 > c] = P[X_1^2 > \frac{c}{c_0}] +$$

$$\left(\frac{-c}{2c_0 \pi x}\right)^{1/2} \sum_{\ell=0}^{k-1} \frac{P[X_{1+2\ell}^2 < -2x]}{\ell! x^\ell (\Gamma(\ell + \frac{1}{2}))^{-1}} P[X_{k-2\ell}^2 > \frac{c}{c_1}] .$$

(b) if  $c_1 < c_0$ , then for

$$\alpha_\ell = \frac{\Gamma(\ell + \frac{1}{2})}{\ell!} \left(\frac{c}{2c_1 x}\right)^\ell \left(\frac{c}{2c_0 \pi x}\right)^{1/2} ,$$

$$P[c_0 X_1^2 + c_1 X_{2k}^2 > c] = P[X_1^2 > \frac{c}{c_0}] +$$

$$e^x \sum_{\ell=0}^{k-1} P[X_{2(k-\ell)}^2 > \frac{c}{c_1}] \cdot \alpha_\ell \left\{ \frac{\mathcal{D}(x^{1/2})}{\sqrt{\pi/4}} - \sum_{t=0}^{\ell-1} \frac{(-1)^t x^{t+1/2}}{\Gamma(t + \frac{3}{2})} \right\} .$$

Examples (for Corollary 3.4). Assume  $c_1 < c_0$ . Then

$$(a) \quad P[c_0 X_1^2 + c_1 X_2^2 > c] = P[X_1^2 > \frac{c}{c_0}] +$$

$$+ e^{-c/2c_0} \left(\frac{4c_1}{\pi(c_0 - c_1)}\right)^{1/2} \mathcal{D}\left(\left(\frac{c}{2} (c_1^{-1} - c_0^{-1})\right)^{1/2}\right) .$$

$$(b) \quad P[c_0 X_1^2 + c_1 X_4^2 > c] = P[X_1^2 > \frac{c}{c_0}] +$$

$$+ e^{-c/2c_0} \left(\frac{c}{2c_0 \pi x}\right)^{1/2} \left[ 2\mathcal{D}(x^{1/2}) \left(1 + \frac{c}{4c_1} (1+x^{-1})\right) - \frac{c}{2c_1 (x)^{1/2}} \right]$$

where  $x = \frac{c}{2} (c_1^{-1} - c_0^{-1})$ .

If both  $m$  and  $n$  are odd, then Theorem 2.1 implies that

$$P\{Q > c\} = \sum_{\ell=1}^{(m-1)/2} \alpha_{1,\ell} P[c_1 X_{1+2\ell}^2 + c_2 X_1^2 > c] + \\ + \sum_{\ell=1}^{(n-1)/2} \alpha_{2,\ell} P[c_1 X_1^2 + c_2 X_{1+2\ell}^2 > c]$$

where

$$\alpha_{1, \frac{m-1}{2}} = \left(\frac{c_1}{c_1 - c_2}\right)^{(m-1)/2}, \quad \alpha_{2, \frac{n-1}{2}} = \left(\frac{c_2}{c_2 - c_1}\right)^{(n-1)/2};$$

$$\alpha_{1, \frac{m-1}{2} - \ell} = \alpha_{1, \frac{m-1}{2}} \cdot \frac{(n-1)}{2\ell} \cdot \left(\frac{c_2}{c_2 - c_1}\right)^\ell, \quad \ell = 1, \dots, \frac{m-3}{2};$$

$$\alpha_{2, \frac{n-1}{2} - \ell} = \alpha_{2, \frac{n-1}{2}} \cdot \frac{(m-1)}{2\ell} \cdot \left(\frac{c_1}{c_1 - c_2}\right)^\ell, \quad \ell = 1, \dots, \frac{n-3}{2}.$$

The next theorem evaluates the probabilities in the summations.

**Theorem 3.5.** Let  $m$  be an odd integer larger than one. For independent chi-square variables and positive  $c_0, c_1$  and  $c$ ,

$$P[c_0 X_m^2 + c_1 X_1^2 > c] = P[c_0^{(0)} X_1^2 + c_1^{(1)} X_1^2 > c] + \\ + \sqrt{\frac{\pi d_1}{d_0}} e^{-(d_0 + d_1)(m-1)/2} \sum_{j=1}^{\infty} \frac{d_0^j j!}{\Gamma(j + \frac{1}{2})} \sum_{k=0}^j \frac{2^{-k}}{(j-k)!} \sum_{\ell=0}^k \frac{I_{2\ell-k}(d_1 - d_0)}{\ell! (k-\ell)!}$$

where  $d_i = c/(4c_i)$ ,  $i = 0, 1$ , and  $I_n(t)$  is the modified Bessel function.

**Remark:** Note that  $I_{-n}(t) = I_n(t)$  and  $I_n(-t) = (-1)^n I_n(t)$ . Using the recursion formula

$$I_{n-1}(t) = I_{n+1}(t) + \frac{2n}{t} I_n(t)$$

all the Bessel functions may be written in terms of  $I_0(d_1-d_0)$  and  $I_1(d_1-d_0)$ .

Proof. Corollary 2.4 implies that

$$P[c_0 X_m^2 + c_1 X_1^2 > c] = P[c_0^{(0)} X_1^2 + c_1^{(1)} X_1^2 > c] + (2c_0)^{(m-3)/2} \sum_{j=0}^{(m-3)/2} f_{Q_{3+2j}}(c)$$

where  $f_{Q_k}(c)$  is the density of

$$Q_k = c_0 X_k^2 + c_1 X_1^2.$$

Theorem 3.1 implies that

$$\begin{aligned} f_{Q_{3+2j}}(c) &= \frac{c^{\frac{3+2j+1}{2}-1} e^{-c/2c_0} {}_1F_1\left(\frac{1}{2}, \frac{3+2j+1}{2}; (c_0^{-1}-c_1^{-1})\frac{c}{2}\right)}{\Gamma\left(\frac{3+2j+1}{2}\right) (2c_0)^{\frac{3+2j}{2}} (2c_1)^{1/2}} \\ &= \frac{c^{j+1} e^{-c/2c_0} {}_1F_1\left(\frac{1}{2}, j+2; (c_0^{-1}-c_1^{-1})\frac{c}{2}\right)}{(j+1)! (2c_0)^{j+3/2} (2c_1)^{1/2}}. \end{aligned}$$

Kummer's transform for confluent hypergeometric functions implies

$$f_{Q_{3+2j}}(c) = \frac{e^{-c/2c_1} c_1^{j+1} {}_1F_1((j+1) + \frac{1}{2}, (j+1)+1, (c_1^{-1} - c_0^{-1}) \frac{c}{2})}{(j+1)! (2c_0)^{(j+1)+1/2} (2c_1)^{1/2}}.$$

Thus  $(2c_0)^{\sum_{j=0}^{(m-3)/2} f_{Q_{3+2j}}(c)}$  is

$$e^{-c/2c_1} \left(\frac{c_0}{c_1}\right)^{1/2} \sum_{k=1}^{(m-1)/2} \frac{\left(\frac{c}{2c_0}\right)^k}{k!} {}_1F_1(k + \frac{1}{2}, k+1; (c_1^{-1} - c_0^{-1}) \frac{c}{2}).$$

Lemma 2 of the Appendix implies

$$(2c_0)^{\sum_{j=0}^{(m-3)/2} f_{Q_{3+2j}}(c)} = \left(\frac{c_0}{c_1}\right)^{1/2} e^{-c/4(c_0^{-1} + c_1^{-1})}.$$

$$\cdot \sum_{k=1}^{(m-1)/2} \frac{\left(\frac{c}{4c_0}\right)^k k!}{\Gamma(k + \frac{1}{2})} \sum_{j=0}^k \frac{2^{-j}}{(k-j)!} \sum_{\ell=0}^j \frac{I_{2\ell-1}(\frac{c}{4}(c_1^{-1} - c_0^{-1}))}{\ell! (j-\ell)!}.$$

Adding this to  $P[c_0^{(0)} \chi_1^2 + c_1^{(1)} \chi_1^2 > c]$  gives

$$P[c_0 \chi_m^2 + c_1 \chi_1^2 > c] \text{ for } d_i = \frac{c}{4c_i}, i = 0, 1. \quad ||$$

Examples (for Theorem 3.5).

Set  $d_i = c/(4c_i)$ ,  $i = 0, 1$ .

$$(a) \quad P[c_0 \chi_3^2 + c_1 \chi_1^2 > c] = P[c_0^{(1)} \chi_1^2 + c_1^{(2)} \chi_1^2 > c] +$$

$$+ \sqrt{4d_0 d_1} e^{-(d_0 + d_1)} (I_0(d_1 - d_0) + I_1(d_1 - d_0)).$$

$$\begin{aligned}
(b) \quad P[c_0 x_5^2 + c_1 x_1^2 > c] &= P[c_0^{(1)} x_1^2 + c_1^{(2)} x_1^2 > c] + \\
&+ \sqrt{4d_0 d_1} e^{-(d_0 + d_1)} \{I_0(d_1 - d_0)[1 + d_0] + \\
&+ I_1(d_1 - d_0)[1 + \frac{4}{3} d_0] + I_2(d_1 - d_0)[\frac{d_0}{3}]\} \\
&= P[c_0^{(1)} x_1^2 + c_1^{(2)} x_1^2 > c] + 4 \sqrt{d_0 d_1} e^{-(d_0 + d_1)} \\
&\quad \{I_0(d_1 - d_0)[1 + \frac{4}{3} d_0] + I_1(d_1 - d_0)[1 + \frac{2d_0}{3} (2 - \frac{1}{d_1 - d_0})]\}
\end{aligned}$$

#### SECTION 4

##### THE DISTRIBUTION OF THE TRACE OF A 2-DIMENSIONAL WISHART MATRIX

The distribution of the trace of a Wishart matrix arises naturally in problems that involve a normal sample covariance matrix. Let  $X_i$ ,  $i = 1, \dots, N$  be a random sample of  $p$ -dimensional vectors from the normal population  $N(\mu, \Sigma)$ . For  $\bar{X} = \sum_{i=1}^N X_i / N$ , it is well known that the sample covariance matrix

$$S = \sum_{j=1}^N (X_j - \bar{X})(X_j - \bar{X})^t (N-1)^{-1}$$

has a Wishart distribution  $W_p(N-1, (N-1)^{-1}\Sigma)$ . (See Anderson (1958).) Pillai and Young (1973) have shown that  $\text{tr}(S)$  has the distribution

$$\sum_{i=1}^p c_i^{(i)} \chi_{N-1}^2$$

where the  $c_i$  are the characteristic roots of the positive definite matrix  $(N-1)^{-1}\Sigma$  and the  $\chi_{N-1}^2$  are independent chi-squared variables each with  $N-1$  degrees of freedom.

In this section, the distribution of  $\text{tr}(S)$  is analyzed for  $p = 2$ .

Theorem 4.1. Let the chi-square variables be independent and  $c_1, c_2$  and  $c$  positive. For an odd integer  $m$  larger than one,

$$\begin{aligned}
P[c_1^{(1)} \chi_m^2 + c_2^{(2)} \chi_m^2 > c] &= P[c_1^{(1)} \chi_{m-2}^2 + c_2^{(2)} \chi_{m-2}^2 > c] \\
&+ \frac{2e^{-(d_1+d_2)} (4d_1 d_2)^{(m-2)/2}}{\Gamma(\frac{m+1}{2})^{-1} \Gamma(m) (\frac{d_1-d_2}{2})^{(m-3)/2}} \{ I_{(\frac{m-3}{2})}^{(d_1-d_2)} \\
&+ \frac{[d_1+d_2]}{[d_1-d_2]} I_{(\frac{m-1}{2})}^{(d_1-d_2)} \},
\end{aligned}$$

where  $d_i = c/(4c_i)$ ,  $i = 1, 2$ .

Proof. Two applications of Theorem 3.2 (first to  $c_1$  and then to  $c_2$ ) yield

$$\begin{aligned}
P[c_1^{(1)} \chi_m^2 + c_2^{(2)} \chi_m^2 > c] &= P[c_1^{(1)} \chi_{m-2}^2 + c_2^{(2)} \chi_{m-2}^2 > c] \\
&+ \frac{c_1 (\frac{c}{2})^{m-1} e^{-c/2c_1}}{\Gamma(m) (c_1 c_2)^{m/2}} {}_1F_1(\frac{m}{2}, m; (c_1^{-1} - c_2^{-1}) \frac{c}{2}) \\
&+ \frac{c_2 (\frac{c}{2})^{m-2} e^{-c/2c_1}}{\Gamma(m) (c_1 c_2)^{(m-2)/2} c_2} {}_1F_1(\frac{m}{2} - 1, m-1; \frac{c}{2} (c_2^{-1} - c_1^{-1})).
\end{aligned}$$

Setting  $d_i = c/(4c_i)$ ,  $i = 1, 2$ , we may write

$$\begin{aligned}
(*) &= P[c_1^{(1)} \chi_m^2 + c_2^{(2)} \chi_m^2 > c] - P[c_1^{(1)} \chi_{m-2}^2 + c_2^{(2)} \chi_{m-2}^2 > c] \\
&= \frac{(2d_1)^{m/2-1} (2d_2)^{m/2}}{\Gamma(m)} e^{-2d_1} {}_1F_1(\frac{m}{2}, m; 2(d_1 - d_2))
\end{aligned}$$



$$+ \frac{(2d_1)^{m/2-1} (2d_2)^{m/2-1}}{\Gamma(m-1)} e^{-2d_2} {}_1F_1\left(\frac{m}{2}-1, m-1; 2(d_2-d_1)\right) .$$

Kummer's transform implies

$$\begin{aligned} e^{-2d_2} {}_1F_1\left(\frac{m}{2}-1, m-1; 2(d_2-d_1)\right) &= e^{-2d_1} {}_1F_1\left(\frac{m}{2}, m-1; 2(d_1-d_2)\right) \\ &= e^{-2d_1} \left\{ {}_1F_1\left(\frac{m}{2}-1, m-2; 2(d_1-d_2)\right) + \frac{(d_1-d_2)}{(m-1)} {}_1F_1\left(\frac{m}{2}, m; 2(d_1-d_2)\right) \right\} \end{aligned}$$

by Lemma 3 of the Appendix. Substitute this in (\*) and write

$$\begin{aligned} (*) &= \frac{e^{-2d_1}}{\Gamma(m)} (2d_1)^{m/2-1} (2d_2)^{m/2-1} \\ &\quad \left\{ (2d_2) {}_1F_1\left(\frac{m}{2}, m; 2(d_1-d_2)\right) + (m-1) \left\{ {}_1F_1\left(\frac{m}{2}-1, m-2; 2(d_1-d_2)\right) \right. \right. \\ &\quad \left. \left. + \frac{(d_1-d_2)}{(m-1)} {}_1F_1\left(\frac{m}{2}, m; 2(d_1-d_2)\right) \right\} \right\} \\ &= \frac{e^{-2d_1} (4d_1 d_2)^{m/2-1}}{\Gamma(m)} \left\{ (d_1+d_2) {}_1F_1\left(\frac{m}{2}, m; 2(d_1-d_2)\right) \right. \\ &\quad \left. + (m-1) {}_1F_1\left(\frac{m}{2}-1, m-2; 2(d_1-d_2)\right) \right\} . \end{aligned}$$

Equation 12.6.3, p. 509 of Abramowitz and Stegun implies that

$${}_1F_1\left(\frac{m}{2}, m; 2(d_1-d_2)\right) = \Gamma\left(\frac{m+1}{2}\right) e^{d_1-d_2} \left(\frac{d_1-d_2}{2}\right)^{-(m-1)/2} I_{\left(\frac{m-1}{2}\right)}(d_1-d_2)$$

and

$${}_1F_1\left(\frac{m}{2}-1, m-2; 2(d_1-d_2)\right) = \Gamma\left(\frac{m-1}{2}\right) e^{d_1-d_2} \left(\frac{d_1-d_2}{2}\right)^{-(m-3)/2} I_{\left(\frac{m-3}{2}\right)}(d_1-d_2) .$$

Substituting these in (\*) we have

$$(*) = \frac{e^{-(d_1+d_2)} \Gamma(\frac{m+1}{2}) (4d_1 d_2)^{m/2-1}}{\Gamma(m) (\frac{d_1-d_2}{2})^{m-3/2}} 2 \left\{ \frac{(d_1+d_2)}{(d_1-d_2)} I_{\frac{m-1}{2}}(d_1-d_2) + I_{\frac{m-3}{2}}(d_1-d_2) \right\} . \parallel$$

Corollary 4.2. Let  $m$  be an odd positive integer larger than one. If  $c_1, c_2$  and  $c$  are positive and the chi-square variables are independent then

$$P[c_1^{(1)} \chi_m^2 + c_2^{(2)} \chi_m^2 > c] = P[c_1^{(1)} \chi_1^2 + c_1^{(2)} \chi_1^2 > c]$$

$$+ \frac{e^{-(d_1+d_2)}}{2(d_1 d_2)^{1/2}} \sum_{j=1}^{\frac{m-1}{2}} \frac{j!}{(2j)!} \left( \frac{8d_1 d_2}{d_1-d_2} \right)^j \{ (d_1-d_2) I_{j-1}(d_1-d_2) + (d_1+d_2) I_j(d_1-d_2) \}$$

where  $d_i = c/(4c_i), i = 1, 2$ .

Examples (for Corollary 4.2).

$$(a) P[c_1^{(1)} \chi_3^2 + c_2^{(2)} \chi_3^2 > c] = P[c_1^{(1)} \chi_1^2 + c_2^{(2)} \chi_1^2 > c]$$

$$+ e^{-(d_1+d_2)} (4d_1 d_2)^{1/2} \left\{ I_0(d_1-d_2) + \frac{(d_1+d_2)}{(d_1-d_2)} I_1(d_1-d_2) \right\} .$$

$$(b) P[c_1^{(1)} \chi_5^2 + c_2^{(2)} \chi_5^2 > c] = P[c_1^{(1)} \chi_1^2 + c_2^{(2)} \chi_1^2 > c]$$

$$+ e^{-(d_1+d_2)} (4d_1 d_2)^{1/2} \left\{ I_0(d_1-d_2) + \frac{I_1(d_1-d_2)}{(d_1-d_2)} [d_1+d_2 + \frac{8}{3} d_1 d_2] \right\}$$

$$\begin{aligned}
& + \frac{I_2(d_1-d_2)}{(d_1-d_2)^2} \left[ \frac{8}{3} d_1 d_2 \right] (d_1+d_2) \} \\
= & P[c_1^{(1)} \chi_1^2 + c_2^{(2)} \chi_1^2 > c] + e^{-(d_1+d_2)} (4d_1 d_2)^{1/2} \{ I_0(d_1-d_2) \\
& [1 + \frac{8}{3} \frac{d_1 d_2 (d_1+d_2)}{(d_1-d_2)^2}] + \frac{I_1(d_1-d_2)}{(d_1-d_2)} [d_1+d_2 + \frac{8}{3} d_1 d_2 (1 - \frac{2(d_1+d_2)}{(d_1-d_2)^2})] \} .
\end{aligned}$$

## SECTION 5

### APPLICATIONS TO CHI-SQUARE TESTS

When sample maximum likelihood estimates are used in chi-square tests for goodness of fit, Chernoff and Lehmann (1954) have shown that the asymptotic distribution of the test has the form

$$Q^* = \sum_{i=1}^r c_i^{(1)} \chi_1^2 + \chi_p^2$$

where the  $c_i$  satisfy  $0 < c_i < 1$  and are distinct and the chi-square variables are independent. For  $p = 2$ , Corollary 5.2 gives the exact percentage points of  $Q^*$  in terms of the distribution of  $\sum_{i=1}^r c_i^{(1)} \chi_1^2$  which may be taken from the tables of Solomon when  $r = 2$  or  $3$  or the tables of Johnson and Kotz when  $r = 4$  or  $5$ . See also the formula of Kleinecke given in the first part of Section 3. For  $p = 4$ , and  $r = 2$ , Corollary 5.4 gives the distribution of  $Q^*$  in terms of Solomon's tables and modified Bessel functions. When  $p = 4$  and  $r = 3$ , Corollary 5.3 gives the distribution of  $Q^*$  in terms of Solomon's tables and Johnson and Kotz's tables. Moore and Spruill (1975) give similar limiting distributions for general chi-squared tests of fit as well as a unified approach to the problem. An example of Moore (1978) for a chi-square test of fit with random cell boundaries and  $M$  cells gives a limiting distribution of the form

$$Q = \chi_{(M-2)}^2 + \lambda \chi_1^2.$$

It is clear that if  $M$  is even, Corollary 3.4 allows one to compute

exact distribution points of  $Q$  from standard chi-square tables. If  $M$  is even Corollary 3.5 implies that the use of Solomon's tables and those for the modified Bessel functions allow the exact computation of percentage points. (If  $M$  is large, it is better to use an approximation.)

The asymptotic distribution of  $(1+n\bar{\tau}_n)$  where  $\bar{\tau}_n$  is the average Kendall tau statistic has been shown by Alvo, Cabilio and Feigen (1982) to be of the form

$$c_1 \chi_{r-1}^2 + c_2 \chi_m^2$$

where  $m = \binom{r-1}{2}$  and  $\frac{c_1}{c_2} = r+1$  and the chi-square variables are independent.

See the results of Le Cam, Mahan and Singh (1983) for other applications.

Theorem 5.1. Assume  $c_0 > c_i > 0$ ,  $i = 1, \dots, n$ , and the chi-square variables are independent. Then

$$\begin{aligned} P\left[\sum_{i=1}^n c_i^{(1)} \chi_{1+c_0}^2 > c\right] &= P\left[\sum_{i=1}^n c_i^{(1)} \chi_1^2 > c\right] \\ &+ K \left\{ P\left[\sum_{i=1}^n c_i^{*(1)} \chi_1^2 < c\right] + \sum_{j=1}^{m-1} \sum_{k=0}^j \left(\frac{c}{2c_0}\right)^j \frac{1}{(j-k)!} \sum_{\substack{\{k_\ell\}_{\ell=1}^n \\ \sum_{\ell=1}^n k_\ell = k \\ 0 \leq k_\ell \leq k}} \left( \prod_{i=1}^n \left(-\frac{2c_i^*}{c}\right)^{k_i} \frac{\Gamma(k_i + \frac{1}{2})}{k_i! \Gamma(\frac{1}{2})} \right) P\left[\sum_{i=1}^n c_i^{*} \chi_{1+2k_i}^2 < c\right] \right\} \end{aligned}$$

where  $c_i^* = (c_i^{-1} - c_0^{-1})^{-1}$  and  $K = e^{-c/2c_0} \prod_{i=1}^n (1 - \frac{c_i}{c_0})^{-1/2}$ .

Proof.

$$\begin{aligned} & P\left[\sum_{i=1}^n c_i^{(i)} \chi_1^2 + c_0 \chi_{2m}^2 > c\right] - P\left[\sum_{i=1}^n c_i^{(i)} \chi_1^2 > c\right] \\ & = (*) = P\left[\sum_{i=1}^n c_i^{(i)} \chi_1^2 < c \text{ and } c_0 \chi_{2m}^2 > c - \sum_{i=1}^n c_i^{(i)} \chi_1^2\right]. \end{aligned}$$

Set  $W' = \sum_{i=1}^n c_i^{*(i)} \chi_1^2$  where  $c_i^* = (c_i^{-1} - c_0^{-1})^{-1}$  ... Because

$$P[\chi_m^2 > a] = e^{-a/2} \sum_{j=0}^{m-1} \frac{(\frac{a}{2})^j}{j!},$$

$$\begin{aligned} (*) & = E\left[I\left(\sum_{i=1}^n c_i^{(i)} \chi_1^2 < c\right) P\left[\chi_{2m}^2 > \frac{c - \sum_{i=1}^n c_i^{(i)} \chi_1^2}{c_0} \middle| \sum_{i=1}^n c_i^{(i)} \chi_1^2 < c\right]\right] \\ & = E\left[I\left(\sum_{i=1}^n c_i^{(i)} \chi_1^2 < c\right) e^{-a/2} \sum_{j=0}^{m-1} \frac{(\frac{a}{2})^j}{j!}\right] \end{aligned}$$

where

$$a = (c - \sum_{i=1}^n c_i^{(i)} \chi_1^2) / c_0.$$

By Lemma 4 of the Appendix, for  $W' = \sum_{i=1}^n c_i^{*(i)} \chi_1^2$  and  $c_i^* = (c_i^{-1} - c_0^{-1})^{-1}$

$$(*) = E\left[I(W' < c) \sum_{j=0}^{m-1} \frac{(c-W')^j}{(2c_0)^j j!}\right] K$$

where

$$K = e^{-c/2c_0} \prod_{i=1}^n \left(1 - \frac{c_i}{c_0}\right)^{-1/2}.$$

Thus

$$\begin{aligned} (*) &= P[W' < c] K + K \sum_{j=1}^{m-1} \frac{(2c_0)^{-j}}{j!} E[I(W' < c) (c-W')^j] \\ &= P[W' < c] K + K \sum_{j=1}^{m-1} \frac{(2c_0)^{-j}}{j!} \sum_{k=0}^j \binom{j}{k} (-1)^k c^{j-k} \\ &\quad \cdot E[I(W' < c) (W')^k]. \end{aligned}$$

Observe that

$$\begin{aligned} E[I(W' < c) (W')^k] &= \sum_{\substack{\{k_\ell\}_{\ell=1}^n \\ \sum_{\ell=1}^n k_\ell = k \\ 0 \leq k_\ell \leq k}} \binom{k}{k_1, \dots, k_n} \\ &\quad \cdot E[I(W' < c) \prod_{i=1}^n (c_i^{*(i)} \chi_1^2)^{k_i}]. \end{aligned}$$

We can note that the desired result is true since

$$\begin{aligned} &E[I(W' < c) \prod_{i=1}^n (c_i^{*(i)} \chi_1^2)^{k_i}] \\ &= \left\{ \prod_{i=1}^n (c_i^{*(i)})^{k_i} \left( \frac{2^{k_i} \Gamma(k_i + \frac{1}{2})}{\Gamma(\frac{1}{2})} \right) \right\} P\left[ \sum_{i=1}^n c_i^{*2} \chi_{1+2k_i}^2 < c \right] \end{aligned}$$

because  $E[(\chi_1^2)^{k_i} h(\chi_1^2)]$  is

$$\frac{\Gamma(k_i + \frac{1}{2})}{\Gamma(\frac{1}{2})} 2^{k_i} E[h(\chi_{1+2k_i}^2)]$$

for any real-valued function  $h$ , i.e.

$$(*) = K \left\{ P[W' < c] + \sum_{j=1}^{m-1} \sum_{k=0}^j \frac{(-c)^{-k}}{(j-k)!} \left(\frac{c}{2c_0}\right)^j \sum_{\substack{\{k_\ell\}_{\ell=1}^n \\ \sum_{i=1}^n k_i = k \\ 0 \leq k_\ell \leq k}} \right\}$$

$$P\left[\sum_{i=1}^n c_i^* \chi_{i+2k_i}^2 < c\right] \left( \prod_{\ell=1}^n \frac{(2c_\ell^*)^{k_\ell}}{\Gamma(\frac{1}{2})} \frac{\Gamma(k_\ell + \frac{1}{2})}{k_\ell!} \right) \Bigg\} . \quad ||$$

This following corollary follows from Theorem 2.6 or Theorem 5.1.

Corollary 5.2. Let  $c_0 > c_i > 0$ ,  $i = 1, \dots, n$  and the chi-square variables are independent. Then

$$P\left[\sum_{i=1}^n c_i^{(i)} \chi_{1+c_0}^2 > c\right] = P\left[\sum_{i=1}^n c_i^{(i)} \chi_1^2 > c\right] + P\left[\sum_{i=1}^n c_i^{*(i)} \chi_1^2 < c\right] K$$

where  $c_i^* = (c_i^{-1} - c_0^{-1})^{-1}$  and  $K = e^{-c/2c_0} \left( \prod_{i=1}^n \left(1 - \frac{c_i}{c_0}\right) \right)^{-1/2}$ .

The following corollary follows directly from Theorem 5.1 for  $m = 2$ :



Corollary 5.3. For  $c_0 > c_i > 0$ ,  $i = 1, \dots, n$  and independent chi-square variables, for

$$K = e^{-c/2c_0} \prod_{i=1}^n \left(1 - \frac{c_i}{c_0}\right)^{-1/2} \quad \text{and} \quad c_i^* = (c_i^{-1} - c_0^{-1})^{-1},$$

$$\begin{aligned} P\left[\sum_{i=1}^n c_i^{(i)} \chi_1^2 + c_0 \chi_4^2 > c\right] &= P\left[\sum_{i=1}^n c_i^{(i)} \chi_1^2 > c\right] + \\ &+ K \left\{ P\left[\sum_{i=1}^n c_i^* \chi_1^2 < c\right] \left(1 + \frac{c}{2c_0}\right) - \sum_{\ell=1}^n \frac{c_\ell^*}{2c_0} \right. \\ &\left. P\left[\sum_{\substack{i=1 \\ i \neq \ell}}^n c_i^* \chi_1^2 + c_\ell^* \chi_3^2 < c\right] \right\}. \end{aligned}$$

In the following corollary, Solomon's tables and the modified Bessel function tables allow one to evaluate the distribution of the quadratic form.

Corollary 5.4. For  $c_0 > c_1 > c_2 > 0$  and independent chi-square variables,

$$\begin{aligned} P[c_0 \chi_4^2 + c_1 \chi_1^2 + c_2 \chi_1^2 > c] &= P[c_1^{(1)} \chi_1^2 + c_2^{(2)} \chi_1^2 > c] \\ &+ \alpha P[c_1^* \chi_1^2 + c_2^* \chi_1^2 < c] + \beta \{ (c_1^* + c_2^*) I_0\left(\frac{c}{4}(c_2^{-1} - c_1^{-1})\right) \\ &- (c_1^* - c_2^*) I_1\left(\frac{c}{4}(c_2^{-1} - c_1^{-1})\right) \} \end{aligned}$$

where  $c_i^* = (c_i^{-1} - c_0^{-1})^{-1}$  and

$$\alpha = e^{-c/2c_0} \left(\frac{c_1^* c_2^*}{c_1 c_2}\right)^{1/2} \left(1 + \frac{c - c_1^* - c_2^*}{2c_0}\right)$$

and

$$\beta = \frac{ce^{-\frac{c}{4}(c_1^{-1} + c_2^{-1})}}{4c_0(c_1c_2)^{1/2}}.$$

Proof. Setting  $K = e^{-c/2c_0} c_0((c_0 - c_1)(c_0 - c_2))^{-1/2}$ , from Corollary 5.3 for  $c_i^* = (c_i^{-1} - c_0^{-1})^{-1}$ ,  $i = 1, 2$ ,

$$\begin{aligned} (*) &= P[c_1^{(1)} \chi_1^2 + c_2^{(2)} \chi_1^2 > c] - P[c_1^{(1)} \chi_1^2 + c_2^{(2)} \chi_1^2 + c_0 \chi_4^2 > c] \\ &\quad + K(1 + \frac{c}{2c_0}) P[c_1^* \chi_1^2 + c_2^* \chi_1^2 < c] \\ &= (2c_0)^{-1} K \{ c_1^* P[c_1^* \chi_1^2 + c_2^* \chi_1^2 < c] + c_2^* P[c_1^* \chi_1^2 + c_2^* \chi_1^2 < c] \}. \end{aligned}$$

By example (a) following Theorem 3.5,

$$\begin{aligned} (*) &= (2c_0)^{-1} K \{ (c_1^* + c_2^*) P[c_1^* \chi_1^2 + c_2^* \chi_1^2 < c] \\ &\quad - e^{-(d_2^* + d_1^*)} \sqrt{4d_1^* d_2^*} ((c_1^* + c_2^*) I_0(d_2^* - d_1^*) \\ &\quad - (c_1^* - c_2^*) I_1(d_2^* - d_1^*)) \} \end{aligned}$$

where  $d_i^* = c/(4c_i^*)$ ,  $i = 1, 2$ .

Because  $K = e^{-c/2c_0} (c_1^* c_2^* d_1 d_2)^{1/2} (\frac{4}{c})$  and  $d_2^* - d_1^* = d_2 - d_1$  and  $d_2^* + d_1^* = d_2 + d_1 - \frac{c}{2c_0}$  and  $\sqrt{4d_1^* d_2^*} = \frac{c}{2(c_1^* c_2^*)^{1/2}}$ , we have

$$(2c_0)^{-1} K \sqrt{4d_2^* d_1^*} e^{-(d_2^* + d_1^*)} = (c_0)^{-1} e^{-(d_1 + d_2)} (d_1 d_2)^{1/2}.$$

Thus

$$\begin{aligned} P[c_1^{(1)} x_1^2 + c_2^{(2)} x_1^2 + c_0 x_4^2 > c] &= P[c_1^{(1)} x_1^2 + c_2^{(2)} x_1^2 > c] \\ &+ KP[c_1^{*(1)} x_1^2 + c_2^{*(2)} x_1^2 < c] \left\{ 1 + \frac{(c - c_1^* - c_2^*)}{2c_0} \right\} \\ &+ \frac{e^{-(d_1 + d_2)}}{c_0} (d_1 d_2)^{1/2} \{ (c_1^* + c_2^*) I_0(d_2 - d_1) - (c_1^* - c_2^*) I_1(d_2 - d_1) \} \end{aligned}$$

$$\text{where } d_i = \frac{c}{4c_i} \text{ and } c_i^* = (c_i^{-1} - c_0^{-1})^{-1}, \quad i = 1, 2. \quad ||$$

# APPENDIX

Lemma 1.

$$\int_0^u x^{\nu-1} (u-x)^{\mu-1} e^{\beta x} dx$$

$$= u^{\mu+\nu-1} \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)} {}_1F_1(\nu, \mu+\nu; \beta u) \text{ for } \mu, \nu \text{ positive.}$$

(See Gradshteyn and Ryzhik (1965) equation 3.383, #1, p. 318.)

Lemma 2. For real-valued  $c$  and positive integer  $n$ ,

$${}_1F_1\left(n + \frac{1}{2}, n+1; c\right) = \frac{(n!)^2}{\Gamma(n + \frac{1}{2})} e^{c/2} \frac{\sqrt{\pi}}{2^n} \sum_{j=0}^n \frac{2^{-j}}{(n-j)!} \sum_{\ell=0}^j \frac{I_{2\ell-j}(\frac{c}{2})}{\ell!(j-\ell)!}.$$

Proof. Equation (13.4.9), p. 507, of Abramowitz and Stegun (1964)

implies that

$${}_1F_1\left(n + \frac{1}{2}, n+1; c\right) = \frac{\Gamma(\frac{1}{2})\Gamma(1+n)}{\Gamma(\frac{1}{2}+n)} \frac{\partial^{(n)}}{\partial c^n} ({}_1F_1(\frac{1}{2}, 1; c)).$$

Because  ${}_1F_1(\frac{1}{2}, 1; c) = e^{c/2} I_0(\frac{c}{2})$ , we have

$$\frac{\partial^{(n)}}{\partial c^n} ({}_1F_1(\frac{1}{2}, 1; c)) = 2^{-n} \frac{\partial^{(n)}}{\partial x^n} (e^x I_0(x)) \Big|_{x=\frac{c}{2}}.$$

The product theorem for differentiation implies that

$$\frac{\partial^{(n)}}{\partial x^n} (e^x I_0(x)) = \sum_{j=0}^n \binom{n}{j} e^x \frac{\partial^{(j)}}{\partial x^j} (I_0(x)).$$

Equation 9.6.29 of Abramowitz and Stegun, p. 376, is

$$\frac{\partial^{(j)}}{\partial x^j} (I_0(x)) = 2^{-j} \sum_{\ell=0}^j \binom{j}{\ell} I_{2\ell-j}(x),$$

making these substitutions,

$$\begin{aligned} {}_1F_1\left(n+\frac{1}{2}, n+1; c\right) &= \frac{\Gamma(\frac{1}{2})\Gamma(1+n)}{\Gamma(\frac{1}{2}+n)} 2^{-n} \sum_{j=0}^n e^{c/2} \binom{n}{j} \\ &\quad 2^{-j} \sum_{\ell=0}^j \binom{j}{\ell} I_{2\ell-j}\left(\frac{c}{2}\right) \\ &= \frac{\sqrt{\pi} (n!)^2}{\Gamma(\frac{1}{2}+n)} e^{c/2} \sum_{j=0}^n \frac{2^{-(n+j)}}{(n-j)!} \sum_{\ell=0}^j \frac{I_{2\ell-j}\left(\frac{c}{2}\right)}{(j-\ell)! \ell!}. \quad || \end{aligned}$$

Lemma 3. Let  $m$  be an odd positive integer larger than one. Then

$${}_1F_1\left(\frac{m}{2}, m-1; t\right) = {}_1F_1\left(\frac{m}{2}-1, m-2; t\right) + \frac{t}{2(m-1)} {}_1F_1\left(\frac{m}{2}, m; t\right).$$

Proof. Equation 13.4.9, p. 507, Abramowitz and Stegun, implies that

$$\begin{aligned} \frac{d}{dt} ({}_1F_1\left(\frac{m}{2}-1, m-2; t\right)) &= \frac{\Gamma(\frac{m}{2})\Gamma(m-2)}{\Gamma(\frac{m}{2}-1)\Gamma(m-1)} {}_1F_1\left(\frac{m}{2}, m-1; t\right) \\ &= \frac{1}{2} {}_1F_1\left(\frac{m}{2}, m-1; t\right). \end{aligned}$$

Thus  ${}_1F_1\left(\frac{m}{2}, m-1; t\right)$  is  $2 \frac{d}{dt} ({}_1F_1\left(\frac{m}{2}-1, m-2; t\right))$  which is

$$2 \frac{d}{dt} \left( \Gamma\left(\frac{m-1}{2}\right) e^{t/2} \left(\frac{t}{4}\right)^{-(m-3)/2} I_{\frac{m-3}{2}}\left(\frac{t}{2}\right) \right)$$

because  ${}_1F_1(k + \frac{1}{2}, 2k+1; 2y) = \Gamma(k+1)e^y(\frac{y}{2})^{-k}I_k(y)$  (from Equation 13.6.3, p. 509, of Abramowitz and Stegun). Thus

$${}_1F_1(\frac{m}{2}, m-1; t) = \Gamma(\frac{m-1}{2})e^{t/2}\left\{\left(\frac{t}{4}\right)^{-(m-3)/2}I_{\frac{(m-3)}{2}}\left(\frac{t}{2}\right) + 2\frac{(m-3)}{2}\frac{d}{dx}(I_\nu(x)x^{-\nu})\Big|_{x=\frac{t}{2}} \text{ and } \nu = \frac{m-3}{2}\right\}.$$

Equation 9.6.28, p. 376, of Abramowitz and Stegun implies that

$$\frac{d}{dx}(I_\nu(x)x^{-\nu}) = x^{-\nu}I_{\nu+1}(x).$$

Thus

$$\begin{aligned} {}_1F_1(\frac{m}{2}, m-1; t) &= \Gamma(\frac{m-1}{2})e^{t/2}\left(\frac{t}{4}\right)^{-(m-3)/2}\left\{I_{\frac{(m-3)}{2}}\left(\frac{t}{2}\right) + I_{\frac{(m-1)}{2}}\left(\frac{t}{2}\right)\right\} \\ &= {}_1F_1(\frac{m}{2}-1, m-2; t) + \frac{t}{2(m-1)}{}_1F_1(\frac{m}{2}, m; t). \quad || \end{aligned}$$

Lemma 4. For  $0 < \gamma < 1$ , and  $h$  a real-valued function,

$$E[h(\chi_m^2)e^{\frac{\gamma}{2}\chi_m^2}] = (1-\gamma)^{-m/2}E[h(\chi_m^2/(1-\gamma))]$$

provided both expectations exist.

Proof.

$$\begin{aligned}
 (*) &= E[h(\chi_m^2) e^{\frac{\gamma}{2} \chi_m^2}] = \int_0^\infty \frac{h(u) u^{m/2-1} e^{-u/2(1-\gamma)} du}{\Gamma(\frac{m}{2}) 2^{m/2}} \\
 &= (1-\gamma)^{-m/2} \int_0^\infty h((1-\gamma)^{-1} w) \frac{w^{m/2-1} e^{-w/2} dw}{\Gamma(\frac{m}{2}) 2^{m/2}}
 \end{aligned}$$

where  $w = (1-\gamma)u \Rightarrow u = (1-\gamma)^{-1}w$ .  $\parallel$

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